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Solving Rubik'S Cubes With Group Theory

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1 Introduction

One of the most noted 3D puzzle inventions is that of the Rubik's cube (Fig. 1), courtesy of the architect Ern˝o Rubik's; through the application of groups, subgroups, cyclic structures, and permutation groups principles.

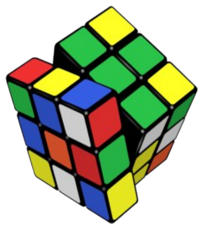


Figure 1 A typical Rubik's cube

Group theory can successfully be used to construct incredibly captivating argumenta- tive tools on Rubik's's Cubes, from a hybrid of theoretical and didactical aspects, to a real teaching instrument. This way, besides learning the subgroup, homomorphism, and equivalence through everyday objects and examples, with the help of those puzzles, these concepts become easier to grasp [1]. The method by which group theory can be embodied in a classroom setting is shown in an earlier study using the concept of a Rubik's Cube. Through hands-on learning, the concepts of group theory have been shown to be not only theoretical but practical in the execution of tasks [2].

The algebra is about looking, in general, at classes of sets with properties and structures related to rings and fields, those elements in the group theory that have connections among them. It amuses me to think about mathematics which can be used also in solving the most interesting problems out there - the Rubik's cube which I used to play with in my childhood. I did not know what makes Rubik's cubes so special until I was at the summer math camp in North Carolina and had a lesson on group theory to show me the processes through the toys. The magic and fascination of mathematics became very interesting to me when I realized that it is connected to the theory of groups and it is more interesting that the whole world of speedcubers admire it. It is at some point during my practical group theory classes that I noticed the similarities among the members of Rubik's cubes. I was able to find a clue about the math I am studying from my real life.

I explore Basics of Group Theory in Sect. 2, the Rubik's Cube and Permutation Groups in Sect. 3, the Rubik's Cube Group in Sect. 4. And I describe some algorithms based on group theory in Sect. 5, other cubes and puzzles in Sect. 6, and conclude in Sect. 7,**2 Basics of Group Theory**

2.1 Groups

A set G equipped with a binary operation $*$ is called a group if it satisfies the following properties:

1. **Closure:** $\forall$ a,b $\in$ G, a $*$ b $\in$ G.
2. **Associativity:** $\forall$ a,b,c $\in$ G, (a $*$ b) $*$ c = a $*$ (b $*$ c).
3. **Identity Element:** $\exists$ e, $\forall$ a $\in$ G, a $*$ e = e $*$ a = a.
4. **Inverses:** $\forall$ a $\in$ G, $\exists$ a^{-1} $\in$ G, a $*$ a^{-1} = a^{-1} $*$ a = e.

2.2 Subgroups

Given a group (G, $*$), a subset H of G is called a subgroup of G if H is also a group under the operation $*$. This relationship is often denoted by H $\leqslant$ G, which is read as "H is a subgroup of G" .

2.2.1 subgroup properties

1. The identity of a subgroup is the identity of the group. We denote it as $e_H = e_G$
2. The inverse of an element in a subgroup is the inverse of the element in the group.
3. The subgroup H is also a group under operation $*$.

2.2.2 normal subgroups

A subgroup N of a group G is said to be a normal subgroup of G if:

$\forall g \in G, \forall n \in N, gng^{-1} \in N$

2.3 Cyclic Groups

A group G is cyclic if:

$\exists g \in G, \forall a \in G, a = g^n , n \in N$

The element g is called a generator of G.

2.4 Cosets

Let H be a subgroup of the group G whose operation is written multiplicatively. Given an element g of G, the left cosets of H in G are the sets obtained by multiplying each element of H by a fixed element g of G (where g is the left factor). In symbols these are,

gH = {gh | h $\in$ H} for g $\in$ G. Similarly, the right cosets are defined as follows:

Hg = {hg | h $\in$ H} for g $\in$ G.

As g varies through the group, it would appear that many cosets (right or left) would be generated. Nevertheless, it turns out that any two left cosets (respectively, right cosets) are either disjoint or identical as sets.

2.5 Symmetric Groups

Let X = {1, 2,..., n} be a set with n elements. The symmetric group on n elements, denoted S_n , is the group of all bijective functions (permutations) from X to itself, under the operation of function composition. Mathematically, this is expressed as:

$S_n = \{\sigma : X \rightarrow X \mid \sigma \text{ is bijective}\}$

where σ is a permutation (a bijective function) in the set X . [1].

3 The Rubik's Cube and Permutation Groups

The Rubik's Cube is made of 26 small cubes, with 14 fixed centers, 12 movable edges, and 8 corners. We define each rotation (move/operation) as a certain permutation of these smaller cubes. These permutations have some special math properties within the field of group theory.

3.1 Permutation Representation

From a single move to the last one, a Rubik's's Cube is made up of 54 plates with different symbols or colors. Permutation is a bijective function that changes the initial structure of the set elements [2].

4 The Rubik's Cube Group

Moving on to the Rubik's Cube Group, the Rubik's Cube group G is formed by six face rotations: R (right), L (left), U (up), D (down), F (front), and B (back). These rotations follow rules that reflect the constraints of the cube. For example, rotating a face four times brings the cube back to its original position, leading to an equation like the following:

$R4 = e$,

where e represents the identity element.

When it comes to solving the Rubik's Cube, algorithms based on group theory can be utilized. These algorithms involve combining sequences of moves to achieve the desired state of the cube.

In more formal terms, if we denote the generators by r,l,u,d,f,b, the Rubik's Cube group can be described by the following presentation:

$G = ⟨r,l,u,d,f, b \mid r4 = l4 = u4 = d4 = f4 = b4 = e⟩$

From a single move to the last, a Rubik's Cube is made up of 54 plates with different symbols or colors. Permutation is a bijective function that changes the initial structure of the set elements [3].

5 Solving Algorithms Based on Group Theory

Algorithms for solving the Rubik's Cube may be created by group theory applications. These shortcuts consist of combining sequences of moves that put the cube in the state we want it to. Usually, rotations are sets of commutators and conjugates.

5.1 Commutators and Conjugation

In group theory, commutators and conjugation serve as tools to create permutations. A commutator takes the form of

$[g, h] = g-1h -1gh$,

indicating the lack of commutativity between G and H. Conjugation involves applying

the permutation G, the permutation H followed by g −1, denoted as ghg −1 .

For instance, to switch two edge pieces, one could employ a commutator like: $[R, U] = R -1U -1RU$.

In general, these commutators and conjugates will cycle the cubes through movements. To relocate a segment to a position without affecting other parts of the cube, one should utilize these combinations of commutators and conjugates.

5.2 Cosets and the Group Structure

Next comes an intriguing concept in groups that captures my attention is cosets. Cosets represent classes in a group that consist of all elements multiplied by an element.

Therefore this concept is similar when we talk about the cube theory, in relation to Ru- bik's cubes as the cosets enable us to showcase the cubes that would emerge by manipulating the cube.

For example, let us consider subgroup H of G representing the rotations of a face F. Whenever G is a move that affects this face F, the corresponding coset gH should encompass all states of F after being rotated by G.

However, it is through analyzing all permutations and their positions that we are likely to eventually find a solution to a Rubik's Cube puzzle. By minimizing the number of components discussed and focusing on topics solved in sequence.

5.3 Layer-by-Layer Method

Layer-by-layer method, that is, is the way they were solved as the group theory. Hence, all sides of the cube are to be solved starting from the topmost layer, the middle layer, to the lower layer last. In terms of group theory, a sequence of permutations can be denoted as:

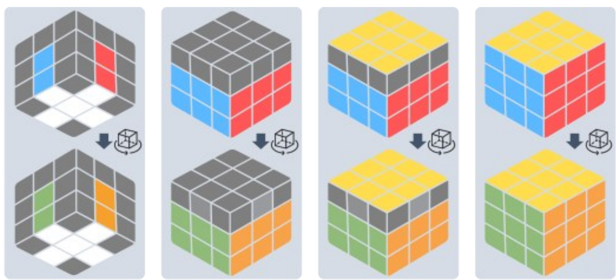


Figure 2 CFOP examples

(a) Cross (b) F2L (c) OLL (d) PLL

$\sigma = \Pi in \sigma i$

where σ is sequence of moves needed to solve the Rubik's Cube, while σi is the each step(or moves) applied for each permutation. The product notation means that these indi- vidual moves are applied sequentially from i = 1 to i = n.

5.4 CFOP

CFOP (a Cross, F2L, OLL, and PLL,see Fig. 3) way consists of a number of steps that differ by the movements of groups [4].

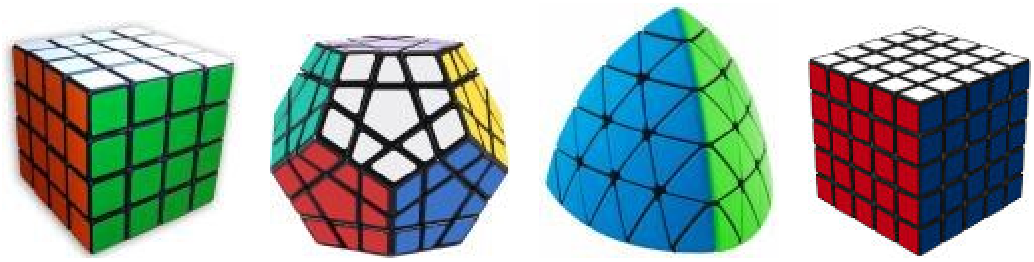
1. **Cross:** Achieving so that the edge elements form a squash on one of the faces. In this approach, you will specify the sequence of changes (permutations) which set the edge pieces correctly, without interfering both the orientation of the whole cube and the rest of its face, see Fig. 2(a).

2. **F2L (First Two Layers):** Simultaneously, putting in pairs of corner and edge pieces completes the first two layers. Similarly, every placement is basically a kind of the group operation, which returns the cube with its cross axis being correct, see Fig. 2(b).

3. **OLL (Orientation of the Last Layer):** The use of the algorithms is in order not violating the system and positioning all the upper layer elements properly. The algorithms consist of a series of

moves that technically rotate the elements on the last layer while keeping the rest of the system in position, see Fig. 2(c).

4. **PLL (Permutation of the Last Layer):** Rearrangement of the positions on the up layer with the method that the given arrangements can be transformed into the original ones. This step requires them to permute the top layer pieces in such a way that the cube solution is perfect, see Fig. 2(d).



(a) 4 ×4×4 (b) 5 ×5×5 (c) Pyraminx (d) Megaminx

Figure 3 Cubes examples

6 Other Cubes and Puzzles

In addition to this creative 3×3×3 Rubik's Cube, we can see some other twisty puzzles like the 2×2×2 cube, 4×4×4 cube and 5×5×5 cube, which are produced after the standard 3×3×3 cube and, in most cases, even better. This unit will take a look at a few of the more prominently known configurations, highlight their distinctions with the canonical 3×3×3, and establish which mathematical procedures are crucial for the success of solving the said configurations.

6.1 4×4×4 Cube (Rubik's Revenge)

The Rubik's Revenge is much harder and much more complex than the original cube owing to an extra layer being added to each side as well as rising the bar to puzzle difficulty [5].

6.1.1 Characteristics

• A 4×4×4 cube consists of 24 side pieces, 24 edges, and 8 little center pieces, see Fig. 3(a).

• After returning the pieces of the puzzle to their proper places, you will realize that no fixed center position can be seen while the puzzle remains solved, thus leaving you in a situation where the center's position of movement is unclear.

6.1.2 Group Theory Application

• Removal of the centers thereof and their rotation may reveal the new group symme- tries, which are intricate center permutations.

• Solving a Rubik's Revenge means transforming it into a cube with 3×3×3 look, as- signing the edges accordingly, and solving the new group of centers afterward.

6.2 5×5×5 Cube (Professor's Cube)

Every student who has ever performed tango with the 5×5×5 cube may have noticed that there was an extra layer added, which made it look much more complicated and multifaceted compared to the previous ones.

6.2.1 Characteristics

• A 5×5×5 cube consists of 25 center pieces, 36 edges, and 8 corner pieces.

• Depth of the cube is a factor influencing the variety of rotations and hence different possible patterns, see Fig. 3(b).

6.2.2 Group Theory Application

The 5×5×5 cube resembles the 4×4×4 cube in a way that somewhat similar processes solve it by fixing the centers and pairing the edges to a 3×3×3 state. When studying the time taken to complete different operations, it is easy to see that the work is done faster with the 5×5×5 cube than with the 4×4×4 cube. The group theory also involves more serious parity errors hidden in the fact that certain pieces on the cube need to be adjusted to their proper orientation through the use of specific algorithms. The proof that there are larger groups of permutations that make mathematics of higher level is a demonstration that the context is equally important for valid observations.

6.3 Pyraminx

The three-dimensional tetrahedral Pyraminx leaves behind the plane structure of the cube and brings to the fore the concept of color grading, using three or four colors for the opposite sides [6].

6.3.1 Characteristics

• Four triangular sides are there, which are further sub-divided.

• Pyraminx consists of 4 corners, 6 edges, and 4 center pieces, see Fig. 3(c).

6.3.2 Group Theory Application

• Pyraminx presents new group structural theories and this relate to the symmetry of a tetrahedron.

• There are several operations in Pyraminx emerging only due to the geometry of tetra- hedral Pyraminx, and thus they have no relation to three coordinates of the cube.

• Group theory gives a sharp tool to understand the rotations and how to solve the brain twister in the most efficient and smart way.

6.4 Megaminx

The megaminx refers to dodecahedron with twelve faces, and each face has five edges which give it a custodian of a completely dissimilar puzzle scenario [7].

6.4.1 Characteristics

• Built from 12 pentagons with one of the moon ' s surfaces divided into pieces, sometimes even with strange shapes.

• A classic Dodecahedron Megaminx consists of 20 corners and 30 edges, see Fig. 3(d).

6.4.2 Group Theory Application

• Cracking the Megaminx is about isolating and tackling the smallest and simplest components of the dodecahedron permutation group.

• Group theory is what prepares the audiences for the resolution of the "puzzle", i.e., reducing the puzzle to a simpler case, as it is accomplished by the Rubik's cube but with more reaching depth.

• Noticing the remarkable structure of the Megaminx (also known as Magic Minx) that includes numerous edges in the form of curves and triangles, the speedcubers use other multiple methods, one of which is called edge pairing.

7 Conclusion

As we wrap up the Rubik's cube may seem small. It holds a charm, especially when looked at through the lens of group theory. Math is fun; that is my motto. By applying group theory principles, like commutators, conjugacy classes, cosets and advanced algorithms to solve Rubik's cubes as groups I find it fascinating how abstract mathematical concepts can address real life challenges.

About the Author

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